AMENABILITY AND EXACTNESS FOR GROUP ACTIONS AND OPERATOR ALGEBRAS

COURSE GIVEN BY CLAIRE ANANTHARAMAN-DELAROCHE

Contents

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Disclaimer

These are notes from a course given by Claire Anantharaman-Delaroche during the minisemester "Amenability beyond groups" at the Erwin Schrödinger Institute in Vienna in March 2007. The reader is asked to bear in mind the informal nature of course notes.

1. Preliminaries

1.1. **Amenability for groups.** Let Γ be a discrete group.

Definition-Proposition 1.1. *Γ is a* amenable *if one of the following equivalent conditions is satisfied:*

(1) $\forall\, \varepsilon > 0\ \forall\, F \subset_{\mathrm{fin}} \Gamma\ \exists\, f \in \ell^1(\Gamma),\ f \geq 0,\ \|f\|_1 = 1\ \max_{s\in F} \|sf - f\| \leq \varepsilon,$

(2) $\forall\, \varepsilon > 0\ \forall\, F \subset_{\mathrm{fin}} \Gamma\ \exists\, f \in \ell^2(\Gamma),\ \|f\|_2 = 1\ \max_{s\in F} \|sf - f\| \leq \varepsilon,$

(3) $\forall\, \varepsilon > 0\ \forall\, F \subset_{\mathrm{fin}} \Gamma$ *there is a positive type function $\varphi : \Gamma \to \mathbb{C}$ with finite support such that* $\max_{s\in F} \big|1 - \varphi(s)\big| \leq \varepsilon$.

Recall that a function $\varphi : \Gamma \to \mathbb{C}$ is *of positive type* if

$$\forall\, n \in \mathbb{N}\ \forall\, s_1, \ldots, s_n\ \forall\, \lambda_1, \ldots, \lambda_n\ \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \varphi(s_i^{-1} s_j) \geq 0.$$

An example of a positive type function is the coefficient function[a] of a representation: let π be a unitary representation of Γ on a Hilbert space H and let $\xi \in H$. The coefficient function is

$$\Gamma \ni t \longmapsto \langle \pi(t)\xi | \xi \rangle\,.$$

Theorem 1.2 (Gelfand, Naimark, Segal)**.** *Every positive type function is a coefficient of a representation. In particular if φ is a positive type function then $\big|\varphi(s)\big| \leq \varphi(e)$ and $\varphi(s^{-1}) = \overline{\varphi(s)}$ for all $s \in \Gamma$.*

1.2. **Amenability for actions.** Let X be a locally compact space and assume that Γ acts on X from the left by homeomorphisms. We shall use the symbol $\Gamma \curvearrowright X$ to denote this situation.

Definition-Proposition 1.3. *The action $\Gamma \curvearrowright X$ is a* amenable *if one of the following equivalent conditions is satisfied:*

(1) $\forall\, \varepsilon > 0\ \forall\, K \Subset X\ \forall\, F \subset_{\mathrm{fin}} \Gamma$ *there is a continuous function $f : X \to \ell^1(\Gamma)_1^+$ (i.e. with values in positive norm one ℓ^1 functions) such that*
$$\max_{(x,s)\in K\times F} \|sf_x - f_{sx}\|_1 \leq \varepsilon,$$

(2) $\forall\, \varepsilon > 0\ \forall\, K \Subset X\ \forall\, F \subset_{\mathrm{fin}} \Gamma$ *there is a continuous function $f : X \to \ell^2(\Gamma)_1^+$ such that*
$$\max_{(x,s)\in K\times F} \|sf_x - f_{sx}\|_2 \leq \varepsilon,$$

(3) $\forall\, \varepsilon > 0\ \forall\, K \Subset X\ \forall\, F \subset_{\mathrm{fin}} \Gamma$ *there is a positive type function $h : X \times \Gamma \to \mathbb{C}$ with compact support such that*
$$\max_{(x,s)\in K\times F} \big|1 - h(x,s)\big| \leq \varepsilon.$$

A function $h : X \times \Gamma \to \mathbb{C}$ is *of positive type* if

$$\forall\, x \in X\ \forall\, n \in \mathbb{N}\ \forall\, s_1, \ldots, s_n\ \forall\, \lambda_1, \ldots, \lambda_n\ \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j h(s_i^{-1}x, s_i^{-1}s_j) \geq 0.$$

Remark 1.4. If Γ is amenable then any action of Γ is amenable, but it is known that the free group $\mathbb{F}_n$ acts amenably on its boundary.

[a]Such functions are also called *matrix elements* of a representation.

1.3. Group algebras.

Definition 1.5. A C*-algebra is a closed $*$-subalgebra of $\mathrm{B}(H)$ for some Hilbert space H.

In particular for any element a of a C*-algebra we have

$$\|a^*a\| = \|a\|^2. \tag{1.1}$$

Theorem 1.6 (Gelfand, Naimark). *Let A be a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. Then there is an isometric isomorphism of A onto a subalgebra of $\mathrm{B}(H)$ for some Hilbert space H.*

Here are some important observations:

(1) If A is a C*-algebra and $A \ni a = a^*$ then $\|a^2\| = \|a\|^2$ and this implies that $\|a\|$ is equal to the spectral radius of a.
(2) If B is another C*-algebra[b] and $\pi : A \to B$ then $\|\pi(a)\| \le \|a\|$ for any $a \in A$and $\pi(A)$ is closed in B. Moreover, if π is injective then it is isometric.

Example 1.7.

(1) $\mathrm{B}(H)$ is a C*-algebra. In particular $M_n(\mathbb{C})$ is a C*-algebra. Finite dimensional C*-algebras are finite products of full matrix algebras.
(2) If X is a locally compact space then $\mathrm{C}_0(X)$ is a C*-algebra. We have

   Theorem 1.8 (Gelfand). *Every Abelian C*-algebra is of the form $C_0(X)$ for a uniquely determined locally compact space X.*

So now let Γ be a discrete group and let σ be a unitary representation of Γ on some Hilbert space H. We can extend σ to a map $\mathbb{C}[\Gamma]$ by

$$\sigma\Big(\sum_{t\in\Gamma} c_t\, t\Big) = \sum_{t\in\Gamma} c_t \sigma(t).$$

We have

$$\Big\|\sigma\Big(\sum_{t\in\Gamma} c_t\, t\Big)\Big\| \le \sum_{t\in\Gamma} |c_t|$$

Definition 1.9. The *full group C*-algebra* of Γ is the completion of $\mathbb{C}[\Gamma]$ in the norm given for $c \in \mathbb{C}[\Gamma]$ by

$$\|c\| = \sup_{\sigma} \big\|\sigma(c)\big\|_{\mathrm{B}(H_\sigma)},$$

where the supremum is taken over all unitary representations of Γ. The full group C*-algebra of Γ is denoted by $\mathrm{C}^*(\Gamma)$.

There is a bijective correspondence between unitary representations of Γ and non degenerate representations of $\mathrm{C}^*(\Gamma)$. The left regular representation $\lambda : \Gamma \to \mathrm{B}\big(\ell^2(\Gamma)\big)$ is injective on $\mathbb{C}[\Gamma] \subset \mathrm{C}^*(\Gamma)$ and this gives a norm on the group algebra. The completion of $\mathbb{C}[\Gamma]$ in this norm is denote by $\mathrm{C}^*_r(\Gamma)$ and is called the *reduced group C*-algebra* of Γ.

Since the norm of $\mathrm{C}^*_r(\Gamma)$ is smaller than that of $\mathrm{C}^*(\Gamma)$, the formel algebra is a quotient of the latter. The left regular representation gives the canonical quotient map $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*_r(\Gamma)$.

When Γ is amenable then every unitary representation of Γ is weakly contained in the regular one. It can be shown that it follows from this that for any σ and any $c \in \mathbb{C}[\Gamma]$ we have

$$\big\|\sigma(c)\big\|_{\mathrm{B}(H_\sigma)} \le \big\|\lambda(c)\big\|_{\mathrm{B}\left(\ell^2(\Gamma)\right)}.$$

Therefore for Γ amenable $\lambda : \mathrm{C}^*(\Gamma) \xrightarrow{\sim} \mathrm{C}^*_r(\Gamma)$.

Theorem 1.10 (Hulanicki). *If $\lambda \mathrm{C}^*(\Gamma) \xrightarrow{\sim} \mathrm{C}^*_r(\Gamma)$ then Γ is amenable.*

Example 1.11. Take $\Gamma = \mathbb{Z}$. Then $\mathrm{C}^*(\Gamma) = \mathrm{C}^*_r(\Gamma) \simeq \mathrm{C}(\mathbb{T})$.

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[b]In fact if B is a Banach $*$-algebra then π must be contractive as well.

1.4. **C*-algebra associated with $\Gamma \curvearrowright X$.** Let $C_0(X)[\Gamma]$ be the vector space of formal (finite) sums

$$\sum_{t\in G} a_t\, t$$

with $a_t \in C_0(X)$. By α we shall denote the action of Γ lifted to $C_0(X)$:

$$\alpha_t(a)(x) = a(t^{-1}x)$$

for all $a \in C_0(X)$, $t \in \Gamma$ and $x \in X$. Now by introducing the following rule of commutation for $a \in C_0(X)$ and $t \in \Gamma$

$$t\,a = \alpha_t(a)t$$

we can multiply elements of $C_0(X)[\Gamma]$. For example

$$(a\,t)(b\,s) = a\alpha_t(b)\,st.$$

$C_0(X)[\Gamma]$ is a $*$-algebra with involution

$$(a\,t)^* = t^{-1}\overline{a} = \alpha_{t^{-1}}(\overline{a})t^{-1}$$

Definition 1.12. A *covariant representation* of $\Gamma \curvearrowright X$ in a Hilbert space H is a pair (π, σ) consisting of a representation π of $C_0(X)$ on H and a unitary representation σ of Γ on H such that

$$\sigma(t)\pi(a)\sigma(t^{-1}) = \pi\big(\alpha_t(a)\big)$$

for all $a \in C_0(X)$ and $t \in \Gamma$.

Any covariant representation of $\Gamma \curvearrowright X$ gives a $*$-homomorphism $\pi \times \sigma : C_0(X)[\Gamma] \to B(H)$

$$(\pi\times\sigma)\Big(\sum_{t\in\Gamma} a_t\, t\Big) = \sum_{t\in\Gamma} \pi(a_t)\sigma(t).$$

Clearly

$$\Big\|(\pi\times\sigma)\Big(\sum_{t\in\Gamma} a_t\, t\Big)\Big\| \le \sum_{t\in\Gamma} \big\|\pi(a_t)\big\|.$$

Definition 1.13. The *full crossed product* of $C_0(X)$ by Γ is the completion of $C_0(X)[\Gamma]$ in the norm

$$\|a\| = \sup_{(\pi,\sigma)} \big\|(\pi\times\sigma)(a)\big\|$$

for any $a \in C_0(X)[\Gamma]$ (we take supremum over all covariant representations. We denote the full crossed product by $C_0(X) \rtimes \Gamma$.

Let us now present the analog of the reduced group C*-algebra associated with $\Gamma \curvearrowright X$. Let π be a representation of $C_0(X)$ on a Hilbert space H_0. Let $H = H_0 \otimes \ell^2(\Gamma) = \ell^2(\Gamma, H_0)$. We can define a covariant representation $\big(\widetilde{\pi}, \widetilde{\lambda}\big)$ of $\Gamma \curvearrowright X$ on H by

$$\big(\widetilde{\pi}(a)\xi\big)(t) = \pi\big(\alpha_{t^{-1}}(a)\big)\xi(t),$$
$$\big(\widetilde{\lambda}_s\xi\big)(t) = \xi(s^{-1}t)$$

for all $\xi \in \ell^1(\Gamma, H_0)$, $a \in C_0(X)$ and all $s, t \in \Gamma$. Now the *reduced crossed product* of $C_0(X)$ by Γ is the completion of $C_0(X)[\Gamma]$ in the norm

$$\|a\| = \sup_{\pi} \big\|\big(\widetilde{\pi}\times\widetilde{\lambda}\big)(a)\big\| \tag{1.2}$$

for all $a \in C_0(X)[\Gamma]$ and the supremum is taken over representations of $C_0(X)$. We denote the reduced crossed product by $C_0(X) \rtimes_r \Gamma$.

It can be shown that the supremum in (1.2) is attained for any injective π.

As in Subsection 1.3 we have the canonical quotient map

$$C_0(X) \rtimes \Gamma \longrightarrow C_0(X) \rtimes_r \Gamma. \tag{1.3}$$

Theorem 1.14. *If the action $\Gamma \curvearrowright X$ is amenable then* (1.3) *is an isomorphism.*

2. Tensor products of C*-algebras

Let A and B be C*-algebras. For simplicity we will assume that both A and B have a unit. There is an obvious $*$-algebra structure on the algebraic tensor product $A \odot B$ and we will now look for C*-norms on $A \odot B$, so that we can complete $A \odot B$ to obtain a C*-algebra.[c]

One way to define such a norm is to embed A and B into $\mathrm{B}(H)$ and $\mathrm{B}(K)$ respectively for some Hilbert spaces H and K. Then $A \odot B \hookrightarrow \mathrm{B}(H \otimes K)$ and the operator norm on $\mathrm{B}(H \otimes K)$ restricted to the image of $A \odot B$ is a C*-norm on $A \odot B$.

Definition 2.1. The minimal tensor product $A \otimes_{\min} B$ of A and B is the completion of $A \odot B$ in the norm

$$\|x\|_{\min} = \sup_{\pi_1, \pi_2} \big\|(\pi_1 \otimes \pi_2)(x)\big\|, \qquad (x \in A \odot B),$$

where the supremum is taken over all representations π_1 and π_2 of A and B respectively.

It is important to note that a result of Takesaki says that the supremum is in fact attained at any pair of faithful representations.

Definition 2.2. The maximal tensor product $A \otimes_{\max} B$ of A and B is the completion of $A \odot B$ in the norm

$$\|x\|_{\max} = \sup_{\pi} \big\|\pi(x)\big\|, \qquad (x \in A \odot B),$$

where the supremum is taken over all representations of $A \odot B$ on Hilbert spaces.

The algebraic tensor product $A \odot B$ is contained both in $A \otimes_{\max} B$ and $A \otimes_{\min} B$, and since $\|\cdot\|_{\max} \geq \|\cdot\|_{\min}$ we have the canonical map $A \otimes_{\max} B \to A \otimes_{\min} B$.

Example 2.3 (Takesaki (1964)). The norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are different on the algebraic tensor product $\mathrm{C}^*_r(\mathbb{F}_2) \odot \mathrm{C}^*_r(\mathbb{F}_2)$.

Exercise 2.4. Let Γ_1 and Γ_2 be discrete groups. Prove that

(1) $\mathrm{C}^*(\Gamma_1 \times \Gamma_2) = \mathrm{C}^*(\Gamma_1) \otimes_{\max} \mathrm{C}^*(\Gamma_2)$,
(2) $\mathrm{C}^*_r(\Gamma_1 \times \Gamma_2) = \mathrm{C}^*_r(\Gamma_1) \otimes_{\min} \mathrm{C}^*_r(\Gamma_2)$.

Solution. Ad (1). Any unitary representation of $\Gamma_1 \times \Gamma_2$ gives us a representation of $\mathbb{C}[\Gamma_1 \times \Gamma_2] = \mathbb{C}[\Gamma_1] \odot \mathbb{C}[\Gamma_2]$ which extends uniquely to a representation of $\mathrm{C}^*(\Gamma_1) \odot \mathrm{C}^*(\Gamma_2)$. Conversely any representation of $\mathrm{C}^*(\Gamma_1) \odot \mathrm{C}^*(\Gamma_2)$ restricts to a representation of $\mathbb{C}[\Gamma_1] \odot \mathbb{C}[\Gamma_2]$ which, again by restriction, gives a unitary representation of $\Gamma_1 \times \Gamma_2$. Now the equality $\mathrm{C}^*(\Gamma_1 \times \Gamma_2) = \mathrm{C}^*(\Gamma_1) \otimes_{\max} \mathrm{C}^*(\Gamma_2)$ follows from the definition of norm on the full group C*-algebra and the norm on the maximal tensor product (Definitions 1.9 and 2.2).

Ad (2). The regular representation of $\Gamma_1 \times \Gamma_2$ acts on $\ell^2(\Gamma_1 \times \Gamma_2) = \ell^2(\Gamma_1) \otimes \ell^2(\Gamma_2)$ and $\lambda(s,t) = \lambda_1(s) \otimes \lambda_2(t)$. Therefore $\mathrm{C}^*_r(\Gamma_1 \times \Gamma_2)$ is by definition of the minimal tensor product equal to $\mathrm{C}^*_r(\Gamma_1) \otimes_{\min} \mathrm{C}^*_r(\Gamma_2)$ (cf. Definition 2.1). □

Example 2.5. The tensor product $M_n(\mathbb{C}) \odot A$ is already a complete space in any tensor norm. Therefore (by uniqueness of the norm on a C*-algebra) we have

$$M_n(\mathbb{C}) \odot A = M_n(\mathbb{C}) \otimes_{\max} A = M_n(\mathbb{C}) \otimes_{\min} A$$

and we write $M_n(\mathbb{C}) \otimes A$ for this C*-algebra.

3. Positive type functions and c.p. maps

Let A and B be (unital) C*-algebras.

Definition 3.1. Let $T : A \to B$ be a linear map. We say that T is positive if it takes positive elements to positive elements. For any $n \in \mathbb{N}$ we have $T_n : M_n(A) \to M_n(B)$ given by

$$T_n(a_{i,j}) = \big(T(a_{i,j})\big).$$

We say that T is completely positive (c.p.) if T_n is positive for all n.

[c]A C*-*norm* is a norm satisfying the C*-identity (1.1).

Lemma 3.2. *A linear map $T : A \to B$ is completely positive if and only if for any $n \in \mathbb{N}$ and any $a_1, \dots, a_n \in A$ the matrix*

$$\big(T(a_i^* a_j)\big) \in M_n(B) \tag{3.1}$$

is positive.

Proof. If T is c.p. then (3.1) is positive because the matrix

$$(a_i^* a_j) = \begin{pmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \tag{3.2}$$

is positive.

Conversely we can show that any matrix $(a_{i,j}) \in M_n(A)_+$ is a finite sum of matrices of the form (3.2). This will end the proof of the lemma.

So let $(a_{i,j}) \in M_n(A)$ be positive. There exists a matrix $(b_{i,j}) \in M_n(A)$ such that

$$(a_{i,j}) = (b_{i,j})^*(b_{i,j})$$

which means that $a_{i,j} = \sum_{k=1}^n b_{k,i}^* b_{k,j}$. In other words

$$(a_{i,j}) = \sum_{k=1}^n (b_{k,i}^* b_{k,j}).$$

□

Corollary 3.3. *Let $B \subset \mathrm{B}(H)$. Then $M_n(B) \subset \mathrm{B}(\mathbb{C}^n \otimes H)$ and we have that $T : A \to B$ is c.p. if and only if for any $n \in \mathbb{N}$, any $a_1, \dots, a_n \in A$ and any $\xi_1, \dots, \xi_n \in H$ we have*

$$\sum_{i,j=1}^n \langle T(a_i^* a_j)\xi_i | \xi_j \rangle \geq 0.$$

Lemma 3.4. *Let Γ be a discrete group.*

(1) *Let $\varphi : \Gamma \to \mathbb{C}$ be a positive type function. Then the map*

$$m_\varphi : \mathbb{C}[\Gamma] \ni c = \sum_{t\in\Gamma} c_t\, t \longmapsto \sum_{t\in\Gamma} \varphi(t) c_t\, t \in \mathbb{C}[\Gamma]$$

extends uniquely to a c.p. map $\Phi : \mathrm{C}_r^*(\Gamma) \to \mathrm{C}_r^*(\Gamma)$.

(2) *Given a c.p. map $\Phi : \mathrm{C}_r^*(\Gamma) \to \mathrm{C}_r^*(\Gamma)$, the function*

$$\varphi : \Gamma \ni s \longmapsto \big\langle \Phi\big(\lambda(s)\big)\lambda(s)^*\delta_e \big| \delta_e \big\rangle \in \mathbb{C} \tag{3.3}$$

is of positive type.

Proof. For simplicity we shall assume that $\varphi(e) = 1$ (in point (1)) and that Φ is unital (in (2)).

Ad (2). Let φ be defined by (3.3) and chose $s_1, \dots, s_n \in \Gamma$. We have $\lambda(s)\delta_e = \delta_s = \rho_{s^{-1}}\delta_e$. Therefore

$$\varphi(s_i^* s_j) = \Big\langle \Phi\big(\lambda(s_i)^*\lambda(s_j)\big)\rho_{S_i^{-1}}\rho_{s_j}\delta_e \Big| \delta_e \Big\rangle = \big\langle \Phi\big(\lambda(s_i)^*\lambda(s_j)\big)\rho_{s_j}\delta_e \big| \rho_{s_i}\delta_e \big\rangle.$$

Now for any $\mu_1, \dots, \mu_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^n \overline{\mu_i}\mu_j \varphi(s_i^{-1}s_j) = \sum_{i,j=1}^n \big\langle \Phi\big(\lambda(s_i)^*\lambda(s_j)\big)\mu_j\rho_{s_j}\delta_e \big| \mu_i\rho_{s_i}\delta_e \big\rangle$$

which is positive because Φ is c.p. (cf. Corollary 3.3).

Ad (1). By Theorem 1.2 we have $\varphi(s) = \langle \pi(s)\xi | \xi \rangle$ for some representation π of Γ on a Hilbert space H and some $\xi \in H$. Define an isometry

$$S : \ell^2(\Gamma) \longrightarrow \ell^2(\Gamma, H)$$

by $(Sf)(s) = f(s)\pi(s)^*\xi$. For any $c \in \mathbb{C}[\Gamma] \subset \mathrm{C}_r^*(\Gamma)$ we have

$$m_\varphi(c) = S^*(c \otimes I_H)S$$

which clearly extends to a c.p. map $C^*_r(\Gamma) \to C^*_r(\Gamma)$. □

Recall that a discrete group Γ is amenable if an only if there is a net (φ_i) of positive type functions with finite support such that $\varphi_i(e) = 1$ for all i and $\varphi_i \xrightarrow[i]{\text{pointwise}} 1$. Since the support of each φ_i is finite, the image of the corresponding map m_φ is finite dimensional (equal to the span of $\operatorname{supp}\varphi_i$ considered as a subset of $\mathbb{C}[\Gamma] \subset C^*_r(\Gamma)$). Moreover, the net (m_i) converges to the identity map of $C^*_r(\Gamma)$ pointwise:

$$\left\| m_{\varphi_i}(a) - a \right\| \longrightarrow 0$$

for any $a \in C^*_r(\Gamma)$.

The property that a C*-algebra A has a net of finite rank c.p. maps $A \to A$ converging pointwise to id_A is called the *completely positive approximation property* (c.p.a.p.). We have thus shown that for an amenable discrete group Γ the reduced group C*-algebra $C^*_r(\Gamma)$ has the c.p.a.p.

4. Nuclearity and amenability

As before we assume that all C*-algebras are unital.

Definition 4.1. Let A and B be C*-algebras.

(1) A u.c.p. map $\psi : A \to B$ is called *nuclear* if there exists a net (φ_i) of finite rank u.c.p. maps $A \to B$ such that

$$\left\| \varphi_i(a) - \psi(a) \right\| \longrightarrow 0$$

for any $a \in A$.

(2) A is called *nuclear* if id_A is a nuclear map.

Theorem 4.2 (Lance 1973)**.** *For a discrete group Γ the C*-algebra $C^*_r(\Gamma)$ is nuclear if and only if Γ is amenable.*

Proof. We have already shown the "if" part at the end of Section 3. Assume that $C^*_r(\Gamma)$ is nuclear and let (Φ_i) be a net of finite rank c.p.u. maps such that

$$\left\| m_\Phi(a) - a \right\| \longrightarrow 0$$

for all $a \in C^*_r(\Gamma)$. Let

$$\varphi_i(s) = \left\langle \Phi_i\big(\lambda(s)\big)\lambda(s)^*\delta_e \middle| \delta_e \right\rangle .$$

By Lemma 3.4 (2) all φ_i are of positive type and it is easy to see that $\varphi_i \xrightarrow{pointwise} 1$ (as $\Phi_i\big(\lambda(s)\big) \to \lambda(s)$). However we have not proved amenability of Γ yet because the supports of φ_i might not be finite.

For this we need the following digresion: let $\Phi : C^*_r(\Gamma) \to C^*_r(\Gamma)$ be a finite rank c.p.u. map. Then

$$\Phi(a) = \sum_{i=1}^n f(_i(a)b_i$$

for some $f_1, \dots, f_n \in C^*_r(\Gamma)'$ and $b_1, \dots, b_n \in C^*_r(\Gamma)$. Let

$$\varphi(s) = \sum_{i=1}^n f_i\big(\lambda(s)\big) \left\langle b_i\lambda(s)^*\delta_e \middle| \delta_e \right\rangle .$$

We claim that φ is in $\ell^2(\Gamma)$. Indeed each of the functions

$$s \longmapsto f_i\big(\lambda(s)\big) \left\langle \delta_{s^{-1}} \middle| b_i^*\delta_e \right\rangle$$

is in $\ell^2(\Gamma)$ (its values are Fourier coefficients of a vector in $\ell^2(\Gamma)$. A theorem of Godement says that a positive type function which is in $\ell^2(\Gamma)$ is necessarily a coefficient of the regular representation. In other words it is of the form

$$s \longmapsto \langle \lambda(s)g | g \rangle$$

for some $g \in \ell^2(\Gamma)$. It is therefore approximable pointwise (so almost uniformly – as Γ is discrete) by functions with compact support (we simply approximate g).

By the reasoning above one can approximate the functions (φ_i) by functions with finite support without destroying the property that $\varphi_i \xrightarrow{pointwise} 1$. This proves amenability of Γ. □

The next theorem has a very similar proof.

Theorem 4.3. *Let* $\Gamma \curvearrowright X$ *be an action. Then* $\Gamma \curvearrowright X$ *is amenable if and only if* $C_0(X) \rtimes_r \Gamma$ *is nuclear.*

The class of nuclear C*-algebras contains all commutative C*-algebras, all finite dimensional C*-algebras as well as tensor products and inductive limits of those. It is also closed under taking extensions. However the algebra $B(H)$ is not nuclear.

4.1. **Approximation properties.**

Definition 4.4. Let E be a Banach space.

1. E has the *Grothendieck approximation property* if id_E is a norm limit on compact subsets of a net (T_i) of finite rank operators.
2. E has the *metric approximation property* if it has the Grothendieck approximation property and the net (T_i) can be chosen to consist of contractions.

A nuclear C*-algebra has the metric approximation property.

Theorem 4.5 (Szankowski (1981))**.** $\mathrm{B}(H)$ *does not have the Grothendieck approximation property.*

Theorem 4.6 (Haagerup (1978))**.** *The* C*-*algebra* $C_r^*(\mathbb{F}_2)$ *has the metric approximation property.*

Note that we know from Theorem 4.2 that $C_r^*(\mathbb{F}_2)$ is not nuclear, i.e. it does not have the completely positive approximation property.

Theorem 4.7 (Kirchberg, Choi)**.** *A* C*-*algebra A is nuclear if and only if for any* C*-*algebra B the canonical map* $A \otimes_{\max} B \to A \otimes_{\min} B$ *is injective.*

In particular nuclearity of A is equivalent to the fact that for any B there is a unique C*-norm on $A \odot B$.

It turns out that a subalgebra of a nuclear algebra need not be nuclear.

Example 4.8. Let Γ be a discrete group acting amenably on a *compact* space X (e.g. $\Gamma = \mathbb{F}_2$ and $X = \partial\mathbb{F}_2$ which is topologically a Cantor set, while the action is amenable). Then $C_r^*(\Gamma)$ is not nuclear if Γ is not amenable, but $C_r^*(\Gamma) \hookrightarrow C(X) \rtimes_r \Gamma$ with the last algebra nuclear by Theorem 4.3.

5. Exactness and boundary amenability

Definition 5.1. A C*-algebra A is *exact* (or *nuclearly embeddable*) if there exists a nuclear embedding of A into some C*-algebra D.

Observe that any subalgebra of a nuclear C*-algebra is exact (by composing the embedding with finite rank c.p.u. approximations of the identity).

Theorem 5.2 (Kirchberg)**.** *Any exact* C*-*algebra can be embedded into a nuclear* C*-*algebra.*

In order to explain the terminology introduced above let us turn to the following problem. Let us fix A and consider the functor $B \mapsto A \otimes_{\min} B$. Now if

$$0 \longrightarrow I \longrightarrow B \longrightarrow B/I \longrightarrow 0$$

is an exact sequence of C*-algebras it can happen that

$$0 \longrightarrow A \otimes_{\min} I \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\min} B/I \longrightarrow 0$$

is not exact (in the middle). This means that the functor $B \mapsto A \otimes_{\min} B$ might fail to be exact.

An explicit example of this was given by Simon Wassermann in 1977. He proved that if $I \subset C^*(\mathbb{F}_2)$ is the kernel of the canonical map $C^*(\mathbb{F}_2) \to C_r^*(\mathbb{F}_2)$ then the sequence

$$0 \longrightarrow C^*(\mathbb{F}_2) \otimes_{\min} I \longrightarrow C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) \longrightarrow C^*(\mathbb{F}_2) \otimes_{\min} C_r^*(\mathbb{F}_2) \longrightarrow 0$$

is not exact.

Theorem 5.3 (Kirchberg)**.** *A* C^**-algebra* A *is nuclearly embeddable if and only if the functor* $B \mapsto A \otimes_{\min} B$ *is exact.*

Note that Wassermann's example shows that $C^*(\mathbb{F}_2)$ is not exact, but we have proved that $C^*_r(\mathbb{F}_2)$ is exact (cf. Example 4.8).

Question 5.4. *Let* Γ *be a discrete group. Assume that* $C^*(\Gamma)$ *is exact. Is* Γ *amenable?*

Question 5.4 has been solved positively for many groups (e.g. maximally almost periodic groups), but the general statement remains open.

Definition 5.5. A discrete group Γ is *exact* if $C^*_r(\Gamma)$ is exact.

Example 5.6.
- Free groups are exact,
- hyperbolic groups are exact (Adams, Germain),
- every discrete subgroup of a connected locally compact group is exact,
- Gromov and Ozawa have shown that there exists a non exact group.

Theorem 5.7 (Ozawa, Anantharaman-Delaroche (2000))**.** *Let* Γ *be a discrete group. Then*

$$\Big(C^*_r(\Gamma) \text{ is exact}\Big) \iff \Big(\Gamma \text{ has an amenable action on a compact space}\Big).$$

Proof. The "$\Leftarrow$" part is contained in Example 4.8.

Let us prove the "$\Rightarrow$" part. The first remark is that if a C^*-algebra A embeds nuclearly into some C^*-algebra D then any embedding of A into $B(H)$ is also nuclear.[d] This means that the inclusion $C^*_r(\Gamma) \hookrightarrow B\big(\ell^2(\Gamma)\big)$ is nuclear. Let (ϕ_k) be the net of finite rank u.c.p. maps approximating this inclusion.

Define

$$h_k(s,t) = \big\langle \phi_k\big(\lambda(s)\big)\lambda(s)^*\delta_s \big| \delta_s \big\rangle\,. \tag{5.1}$$

Then $h_k : \Gamma \times \Gamma$ is a continuous and bounded function ($\big|h_k(s,t)\big| \leq 1$ for all s,t). We can extend h_k to a continuous map $\beta\Gamma \times \Gamma \to \mathbb{C}$.

The group Γ acts on $\beta\Gamma$ and it will be enough to show that this action is amenable. To that end recall from Definition-Proposition 1.3 that this action is amenable if and only if there exists a net (h_k) of positive type functions with compact support which converges to 1 uniformly on compact subsets of $\beta\Gamma \times \Gamma$.

We claim that the functions h_k defined by (5.1) are of positive type (cf. text preceeding Remark 1.4).

We will prove that for any $x \in \Gamma$, any $n \in \mathbb{N}$ and any $s_1, \dots, s_n \in \Gamma$ the matrix

$$\big(h_k(s_i^{-1}x, s_i^{-1}s_j)\big)$$

is positive (this will be enough by continuity).

We have

$$h_k(s_i^{-1}x, s_i^{-1}s_j) = \Big\langle \phi_k\big(\lambda(s_i)^*\lambda(s_j)\big)\lambda(s_j)^*\lambda(s_i)\delta_{s_i^{-1}x} \Big| \delta_{s_i^{-1}x} \Big\rangle = \Big\langle \phi_k\big(\lambda(s_i)^*\lambda(s_j)\big)\delta_{s_j^{-1}x} \Big| \delta_{s_i^{-1}x} \Big\rangle\,,$$

so for $\mu_1, \dots, \mu_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^n \overline{\mu_i}\mu_j h_k(s_i^{-1}x, s_i^{-1}s_j) = \sum_{i,j=1}^n \Big\langle \phi_k\big(\lambda(s_i)^*\lambda(s_j)\big)\mu_j\delta_{s_j^{-1}x} \Big| \mu_i\delta_{s_i^{-1}x} \Big\rangle \geq 0$$

because ϕ_k is completely positive (cf. Corollary 3.3).

It is not hard to show that $h_k(s,t) \to 1$ uniformly on compact subsets of $\beta\Gamma \times \Gamma$, but as in the proof of Theorem 4.2 the supports of (h_k) might not be compact. This is a technical point which can be overcome by appropriate approximation. □

[d]This follows from Arveson's theorem which says that $B(H)$ is an injective operator system, which means in this case that if $\imath$ is the injection of A into D and $\jmath$ is the inclusion of A into $B(H)$ then there exists a u.c.p. map $\psi : D \to B(H)$ such that $\psi \circ \imath = \jmath$. In particular, if (Φ_k) is the net of finite rank u.c.p. maps approximating $\imath$ then $\jmath$ can be approximated by finite rank u.c.p. maps $\Phi_k \circ \imath : A \to B(H)$.

The property of admitting an amenable action on a compact space is called *boundary amenability.*

Let us recall from Definition 1.3 (3) that the action of Γ on $\beta\Gamma$ is amenable if

$$\forall\, \varepsilon > 0\ \forall\, E \subset_{\text{fin}} \Gamma\ \exists\, h : \beta\Gamma \times \Gamma \to \mathbb{C},\ \text{of positive type}\ \exists\, E' \subset_{\text{fin}} \Gamma$$

such that

(1) $\sup\limits_{(x,s)\in\beta\Gamma\times E} \big|1 - h(x,s)\big| \leq \varepsilon,$
(2) $\operatorname{supp} h \subset \beta\Gamma \times E'.$

Since h is continuous we can replace (2) with

$$\sup_{(x,s)\in\Gamma\times E} \big|1 - h(x,s)\big| \leq \varepsilon$$

and consider instead $h : \Gamma \times \Gamma \to \mathbb{C}$. The fact that h is of positive type is that for any $n \in \mathbb{N}$, any $x \in \Gamma$ and any $s_1, \dots, s_n \in \Gamma$ the matrix

$$\big(h(s_i^{-1}x, s_i^{-1}s_j\big) \tag{5.2}$$

is positive.

We will now make a change of variables. Let

$$J : \Gamma \times \Gamma \ni (s,t) \longmapsto (s^{-1}, s^{-1}t) \in \Gamma \times \Gamma$$

and let $k = h\circ J$. This means that we have

$$k(s,t) = h\big(s^{-1}, s^{-1}t\big). \tag{5.3}$$

In particular $\operatorname{supp} k$ is in the "strip"

$$\big\{(s,t)\big| s^{-1}t \in E'\big\}.$$

Moreover we have

$$\sup_{s^{-1}t\in E} \big|1 - k(s,t)\big| \leq \varepsilon.$$

and k is a positive kernel in the usual sense, i.e. for any $n \in \mathbb{N}$ and any $s_1, \dots, s_n \in \Gamma$ the matrix

$$\big(k(s_i, s_j)\big)$$

is positive (this is because of (5.3) and positivity of (5.2) with $x = e$).

We have thus shown the following:

Theorem 5.8. *Let Γ be a discrete groups. Then Γ is exact if and only if*

$$\forall\, \varepsilon > 0\ \forall\, E \subset_{\text{fin}} \Gamma\ \exists\, k : \Gamma \times \Gamma \to \mathbb{C},\ \textit{bounded, of positive type}\ \exists\, E' \subset_{\text{fin}} \Gamma$$

such that

(1) $\sup\limits_{s^{-1}t\in E} \big|1 - k(s,t)\big| \leq \varepsilon,$
(2) $\operatorname{supp} k \subset \big\{(s,t)\big| s^{-1}t \in E'\big\}.$

Note that the conditions describing exactness of Γ are expressed without using the group structure of Γ. In fact exactness is a metric concept in the sense described below.

Assume that Γ is finitely generated and consider the metric d given by word length function with respect to some set of generators of Γ. The d is left invariant metric.

Corollary 5.9. Γ *is exact if and only if*

$$\forall\, \varepsilon > 0\ \forall\, R > 0\ \exists\, k : \Gamma \times \Gamma \to \mathbb{C},\ \textit{bounded, of positive type}\ \exists\, R' > 0$$

such that

(1) $\big|1 - k(s,t)\big| \leq \varepsilon$ *if* $d(s,t) \leq R,$
(2) $k(s,t) = 0$ *if* $d(s,t) \geq R'.$

Definition 5.10. Let (X,d) be a countable metric space. We say that (X,d) is *exact* if

$$\forall\,\varepsilon>0\ \forall\,R>0\ \exists\,k:X\times X\to\mathbb{C},\ \text{bounded, of positive type}\ \exists\,R'>0$$

such that

(1) $\left|1-k(x,y)\right|\le\varepsilon$ if $d(x,y)\le R$,
(2) $k(x,y)=0$ if $d(x,y)\ge R'$.

Remark 5.11. A function (kernel) $k:X\times X\to\mathbb{C}$ is of positive type if and only if there exists a Hilbert space $\mathcal{H}$ and a map $\xi:X\ni x\mapsto\xi_x\in\mathcal{H}$ such that $k(x,y)=\langle\xi_x|\xi_y\rangle$.

Proposition 5.12. *(X,d) is exact if*

$$\forall\,\varepsilon>0\ \forall\,R>0\ \exists\,\textit{Hilbert space}\ \mathcal{H}\ \textit{and}\ \xi:X\to\mathcal{H}_1\ \exists\,R'>0$$

such that

(1) $\left|1-\langle\xi_x|\xi_y\rangle\right|\le\varepsilon$ *if* $d(x,y)\le R$,
(2) $\langle\xi_x|\xi_y\rangle=0$ *if* $d(x,y)\ge R'$.

In all of our consideratios we can take the Hilbert space to be real and replace k by its real part (it remains positive type). Then condition (1) of Proposition 5.12 can be raplaced by

$$\|\xi_x-\xi_y\|\le\varepsilon\text{if } d(x,y)\le R$$

because in a real Hilbert space $\|\xi_x-\xi_y\|^1=2\big(1-\langle\xi_x|\xi_y\rangle\big)\big|$.

6. Yu's property (A)

Let (X,d) be a metric space with bounded geometry, i.e. for any $R>0$ there exists $N\in\mathbb{N}$ such that for any $x\in X$ the cardinality $\#B(x,R)$ is smaller than N.

In what follows we shall denote by $\mathcal{F}(S)$ the set of all finite subsets of a set S.

Definition 6.1 (Yu's property (A))**.** (X,d) has *property* (A) if

$$\forall\,\varepsilon>0\ \forall\,R>0\ \exists\,A:X\to\mathcal{F}(X\times N)\ \exists\,R'>0$$

such that

(1) $\dfrac{\#\big(A(x)\triangle A(y)\big)}{\#\big(A(x)\cap A(y)\big)}\le\varepsilon$ if $d(x,y)\le R$,
(2) $\exists\,R'>0\ \forall\,x\in X\ \ A(x)\subset B(x,R')$.

Theorem 6.2 (Higson, Roe (2000))**.** *Yu's property* (A) *is equivalent to exactness (Definition 5.10).*

Sometimes exact groups are called *groups with property* (A).

7. Exactness and uniform embeddability

Definition 7.1. A metric space (X,d) is *uniformly embeddable into a Hilbert space* if there exists a Hilbert space $\mathcal{H}$, a map $f:X\to\mathcal{H}$ and two functions $\rho_1,\rho_2:\mathbb{R}_+\to\mathbb{R}$ such that both ρ_i are non decreasing, $\lim\limits_{t\to\infty}\rho_i(t)=\infty$ and for any $x,y\in X$ we have

$$\rho_1\big(d(x,y)\big)\le\big\|f(x)-f(y)\big\|\le\rho_2\big(d(x,y)\big).$$

Proposition 7.2 (Dădărlat-Guentner)**.** *A metric space (X,d) is uniformly embeddable into a Hilbert space if andonly if*

$$\forall\,\varepsilon>0\ \forall\,R>0\ \exists\,\textit{Hilbert space}\ \mathcal{H}\ \textit{and}\ \xi:X\to\mathcal{H}_1\ \exists\,R'>0$$

such that

(1) $\|\xi_x-\xi_y\|\le\varepsilon$ *if* $d(x,y)\le R$
(2) $\lim\limits_{r\to\infty}\sup\limits_{d(x,y)\ge r}\big|\langle\xi_x|\xi_y\rangle\big|=0.$

Idea of proof (in one direction). Recall a that a function $k:X\times X\to\mathbb{R}$ is a *symmetric kernel conditionally of negative type with zero diagonal* if

- $k(x,x) = 0$ for all $x \in X$,
- $k(x,y) = k(y,x)$ for all $x, y \in X$,
- $\forall\, n \in \mathbb{N}\ \forall\, x_1, \dots, x_n \in X\ \forall\, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j k(x_i, x_j) \leq 0.$$

It is known that a symmetric kernel k with zero diagonal is conditionally of negative type if and only if there exists a real Hilbert space $\mathcal{H}$ and a function $f : X \to \mathcal{H}$ such that $k(x,y) = \big\|f(x) - f(y)\big\|^2$ for all $x, y \in X$.[e]

Theorem 7.3 (Schönberg)**.** *A symmetric kernel k with zero diagonal is conditionally of negative type if and only if for any $t > 0$ the function $(x,t) \mapsto \exp\big(-tk(x,y)\big)$ is of positive type.*

We are now in position to prove the implication "⇒", i.e. that uniform embeddability of (X, d) into a Hilbert space implies exactness.

Let $f : X \to \mathcal{H}$ be a uniform embedding of X into a Hilbert space $\mathcal{H}$ with control functions ρ_1, ρ_2. For $t > 0$ define

$$h_t(x,y) = \exp\big(-t\big\|f(x) - f(y)\big\|^2\big).$$

Take $R > 0$ and $x\,y \in X$ such that $d(x,y) \leq R$. Then since

$$h_t(x,y) = e^{-t\big\|f(x)-x(y)\big\|^2} \leq e^{-t\rho_2\big(d(x,y)\big)^2} \leq e^{-t\rho_2(R)^2}$$

(ρ_2 is non decreasing) we have

$$\big|1 - h_t(x,y)\big| \leq \exp\big(-t\rho_2(R)^2\big). \tag{7.1}$$

We can therefore take now t such that (7.1) is smaller than ε.

On the other hand, if $d(x,y) > r$ then

$$h_t(x,y) \leq \exp\big(-t\rho_1\big(d(x,y)\big)^2\big) \leq \exp\big(-t\rho_1(r)^2\big)$$

and the last term on the right hand side goes to 0 as $r \to \infty$. □

Corollary 7.4 (Yu)**.** *Every metric space with bounded geometry and property* (A) *is uniformly embeddable into a Hilbert space. In particular exact groups are uniformly embeddable into a Hilbert space.*

Let us investigate further the relation between uniform embeddability and exactness. The most suitable definitions of these properties are contained in Proposition 7.2 and Proposition5.12.

Assume that we have a function (kernel) $h : \Gamma \times \Gamma \to \mathbb{R}$. We say that h is Γ-invariant if

$$h(sx, sy) = h(x, y)$$

for all $s, x, y \in \Gamma$. If h is invariant then h is encoded in a function of one variable $\varphi(t) = h(e,t)$. In this case h is a positive type kernel if and only if φ is a function of positive type (simply because $\varphi(s^{-1}t) = h(s,t)$. Also conditions (1) and (2) of Proposition 5.12 read

(1) $\big|1 - \varphi(t)\big| \leq \varepsilon$ if $\ell(t) \leq R$,
(2) $\varphi(t) = 0$ if $\ell(t) \geq R'$,

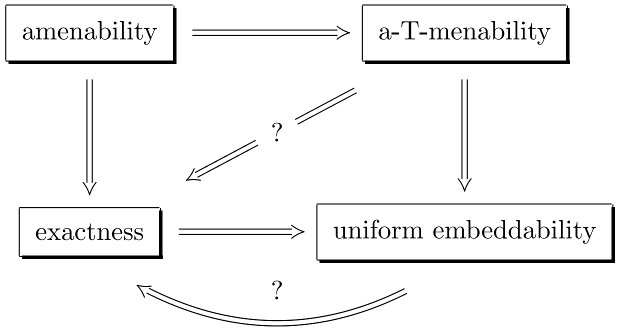
where ℓ is the word length function on Γ. Note that this is nothing but the definition of amenability (cf. Definition-Proposition 1.1 (3)).

In other words the invariant analog of exactness is amenability.

In the same way the invariant analog of uniform embeddability is a-T-menability (then we get $\varphi \in C_0(\Gamma)$ not of compact support).

We have the following diagram of known and unknown relationships between various properties of finitely generated groups (question marks indicate open problems):

[e]Sometimes this statement is referred to as Schönberg's theorem.



Let us mention that Gromov has indirectly shown that there is finitely generated group which is not uniformly embeddable into a Hilbert space.

7.1. **Compression constants and uniform embeddability.** Let $f : \Gamma \to \mathcal{H}$ be a uniform embedding with control functions ρ_1, ρ_3. First we note that ρ_2 can always be taken affine. Indeed if we look at $x, y \in \Gamma$ and take a geodesic path $(x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ from x to y (so the distance between x_i and x_{i+1} is one and $d(x, y) = n$) then

$$\|f(x) - f(y)\| \leq \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| \leq \rho_2(1) d(x, y).$$

Definition 7.5 (Gromov)**.** The *compression function* of a uniform embedding $f : \Gamma \to \mathcal{H}$ is the function $\rho_f : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\rho_f(r) = \inf_{d(x,y) \geq r} \|f(x) - f(y)\|.$$

The compression function is non decreasing. Moreover for any $r \in \mathbb{R}_+$

$$\rho_1(r) \leq \rho_f(r)$$

and

$$\rho_f\big(d(x, y)\big) \leq \|f(x) - f(y)\|$$

for all $x, y \in X$.

Definition 7.6 (Gromov)**.** The *asymptotic compression constant* of a uniform embedding $f : \Gamma \to \mathcal{H}$ is

$$R_f = \sup\{\alpha \geq 0 \,|\, \exists\, a, b > 0 \;\; t^\alpha \leq a\rho_f(t) + b \text{ for all } t > 0\}.$$

The observation at the beginning of this subsection shows that $R_f \in [0, 1]$.

Example 7.7. Take $\Gamma = \mathbb{F}_2$ an let E be the set of edges of the Cayley graph of $\mathbb{F}_2$. We have a uniform embedding $f : \mathbb{F}_2 \to \ell^2(E)$ such that $d(x, y) = \|f(x)\|^2$ for all $x, y \in \mathbb{F}_2$. It is defined as follows: let w be a word in $\mathbb{F}_2$. There is a unique geodesic path from e to w in the Cayley graph of $\mathbb{F}_2$. If this path is $(e_1, \ldots, e_n)$ with $e_i \in E$ we let

$$f(w) = \sum_{i=1}^{n} \delta_{e_i}.$$

In this case $R_f = \frac{1}{2}$.

Example 7.8. If f is a *quasi isometric embedding* (i.e. a uniform embedding with ρ_1 – an affine function) then $R_f = 1$.

Definition 7.9. The Hilbert space compression constan of Γ is

$$R(\Gamma) = \sup_f R_f$$

(the supremum is over all uniform embeddings f of Γ into a Hilbert space).

Clearly $R(\Gamma) \in [0,1]$. We have $R(\mathbb{F}_2) = 1$ and for any $\varepsilon > 0$ a uniform embedding f of $\mathbb{F}_2$ into a Hilbert space such that $R_f \geq 1 - \varepsilon$ can be constructed by a modification of Example 7.7 (Bourgain (1986), Guentner-Kaminker (2004)).

Note that the supremum $R(\mathbb{F}_2) = 1$ is never attained because if it were, the free group would embed quasi isometrically into a Hilbert space. By results of Bourgain (the 3-regular tree is not embeddable quasi isometrically into a Hilbert space) this is not possible.

Theorem 7.10 (Guentner-Kaminker)**.** *If* $R(\Gamma) > \frac{1}{2}$ *then* Γ *is exact.*

Before giving a sketch of the proof of Theorem 7.10 let us mention that In a 2006 paper Arzhentzeva, Drutu and Sapir showed that for any $\alpha \in [0,1]$ there exists an exact group Γ with $R(\Gamma) = \alpha$. For example, for $\Gamma = \mathbb{Z} \wr \mathbb{Z}$, we have $\frac{2}{3} \leq R(\Gamma) \leq \frac{3}{4}$ and $R(\Gamma \wr \mathbb{Z})\frac{1}{2}$. Moreover by iterating the wreath product we can get $R(\Gamma)$ arbitrarily small while $\Gamma = (\cdots(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \cdots) \wr \mathbb{Z}$ is not only exact, but also amenable.

Sketch of proof of Theorem 7.10. The assumption is that there exists $\alpha > 0$, a uniform embedding $f : \Gamma \to \mathcal{H}$ and $r_0 > 0$ such that

$$d(x,y)^{\frac{1}{2}+\alpha} \leq \big\|f(x) - f(y)\big\| \tag{7.2}$$

is $d(x,y) \geq r_0$.

For $t > 0$ consider

$$h_t(x,y) = \exp\bigl(-t\big\|f(x) - f(y)\big\|^2\bigr).$$

h_t is of positive type (cf. proof of Proposition 7.2).

In what follows we will write h for h_1. We need to show that for any $\varepsilon > 0$ there exists a kernel $k : \Gamma \times \Gamma \to \mathbb{R}$ of positive type such that $\|h - k\|_\infty \leq \varepsilon$ and k is of *finite propagation,* i.e. its support is in the "strip" $\bigl\{(x,y)\big| d(x,y) \leq R\bigr\}$.

First let us check that we have

$$\lim_{n\to\infty} \sup_{x\in\Gamma} \Bigl(\sum_{y:\, d(x,y)\geq n} h(x,y) \Bigr) = 0.$$

Indeed, fix $x \in \Gamma$ and let $r \geq r_0$. We have by definition of h and (7.2)

$$\begin{aligned}
\sum_{d(x,y)\geq n} h(x,y) &= \sum_{m\geq n} \sum_{d(x,y)=m} h(x,y) \\
&\leq \sum_{m\geq n} \sum_{d(x,y)=m} \exp\bigl(-d(x,y)^{1+2\alpha}\bigr) \\
&= \sum_{m\geq n} \sum_{d(x,y)=m} \exp\bigl(-m^{1+2\alpha}\bigr).
\end{aligned}$$

Now note that the number of elements of $\bigl\{y\big| d(x,y) = m\bigr\}$ is less or equal than $(\#S)^m$ where S is the symmetric set of generators of Γ giving the word length metric. Therefore

$$\begin{aligned}
\sum_{d(x,y)\geq n} h(x,y) &\leq \sum_{m\geq n} (\#S)^m \exp\bigl(-m^{1+2\alpha}\bigr) \\
&= \sum_{m\geq n} \frac{(\#S)^m}{\exp\bigl(m\, m^{2\alpha}\bigr)} \\
&= \sum_{m\geq n} \left(\frac{\#S}{\exp\bigl(m^{2\alpha}\bigr)} \right)^m \\
&\leq \sum_{m\geq n} \left(\frac{\#S}{\exp\bigl(n^{2\alpha}\bigr)} \right)^m \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

The proof will be finished when we prove that the following lemma:

Lemma 7.11. *Let* $h : \Gamma \times \Gamma \to \mathbb{R}_+$ *be positive type kernel such that*

$$c = \sup_{x \in \Gamma} \sum_{y \in \Gamma} h(x, y) < \infty.$$

and

$$c_n = \sup_{x \in \Gamma} \Big(\sum_{y\colon d(x,y) \geq n} h(x, y) \Big) \xrightarrow[n \to \infty]{} 0. \tag{7.3}$$

Then for any $\varepsilon > 0$ *there exists a positive type kernel* $k : \Gamma \times \Gamma \to \mathbb{R}$ *such that* $\|h - k\|_\infty < \varepsilon$ *and* k *is of finite propagation.*

Proof of Lemma 7.11. The kernel h defines $\operatorname{Op} h \in \mathrm{B}\big(\ell^2(\Gamma)\big)$ by[f]

$$\big((\operatorname{Op} h)\xi\big)(x) = \sum_{y \in \Gamma} h(x, y)\xi(y).$$

One checks that $\operatorname{Op} h$ is bounded and $\|\operatorname{Op} h\| \leq c$. Let h_n be the cut-off of h

$$h_n(x, y) = \begin{cases} 0 & d(x, y) > n, \\ h(x, y) & d(x, y) \leq n. \end{cases}$$

Thus defined h_n is not of positive type, but we have

$$\|\operatorname{Op}(h - h_n)\| \leq c_n. \tag{7.4}$$

Let $\mathrm{C}^*_u(\Gamma)$ be the C^*-algebra of operators on $\ell^2(\Gamma)$ generated by $\operatorname{Op} k$ for all kernels k of finite propagation.[g]

Now $\operatorname{Op} h \in \mathrm{C}^*_u(\Gamma)$ because of (7.4) and (7.3) and $\operatorname{Op} h$ si positive. Let $T = \sqrt{\operatorname{Op} h}$. Then $T \in \mathrm{C}^*_u(\Gamma)$. We can therefore approximate T by kernels with finite propagation. More precisely, for any $\eta > 0$ there exists a finite propagation kernel k' such that

$$\|T - \operatorname{Op} k'\| \leq \eta.$$

Denote $V = \operatorname{Op} k'$ and let

$$k(x, y) = \langle V\delta_x | V\delta_y \rangle$$

Then k is a positive type kernel with finite propagation. Now

$$\big|h(x, y) - k(x, y)\big| = \big|\langle (T^*T - V^*V)\delta_x | \delta_y \rangle\big| \leq \|T^*T - V^*V\|$$

which we can make arbitrarily small. □

□

[f] This is why h is called a "kernel".
[g] This is the Roe algebra; it is isomorphic to $\mathrm{C}(\beta\Gamma) \rtimes_r \Gamma$.